SECTION 7.4: GROWTH MODELS

UNLIMITED (UNINHIBITED) GROWTH:

Underlying principle: the rate of growth is proportional to amount present.

$$\frac{dy}{dt} = k\,y \implies y(t) = y_0 e^{kt}$$

Here, k is the constant of proportionality and $y(0) = y_0$ is the initial amount.

LIMITED (INHIBITED) GROWTH:

Underlying principle: the rate of growth is proportional to the room available to grow.

$$\frac{dy}{dt} = k\,(L - y) \implies y(t) = (y_0 - L)e^{-kt} + L$$

Here, k is the constant of proportionality, y_0 is the initial amount, and L is the limiting amount.

LOGISTIC GROWTH:

Underlying principle: the rate of growth is proportional to the amount present **and** the room available to grow:

$$\frac{dy}{dt} = k\,y\,(L - y) \implies y(t) = \frac{L}{1 + Ce^{-kLt}}$$

Here, k is the constant of proportionality, L is the limiting amount, and $C = \frac{L}{y_0} - 1$ where y_0 is the initial amount.

APPLICATIONS OF EXPONENTIAL FUNCTIONS (FROM COLLEGE ALGEBRA)

COMPOUND INTEREST:

If an initial principal P is invested at an annual rate r and the interest is compounded n times per year, the amount in the account after t years, $A(t)$ is given by

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

CONTINUOUSLY COMPOUNDED INTEREST:

If an initial principal P is invested at an annual rate r and the interest is compounded continuously, the amount in the account after t years, $A(t)$ is given by

$$A(t) = Pe^{rt}$$

UNINHIBTED GROWTH:

If a population increases according to The Law of Uninhibited Growth, the number of organisms at time t, $N(t)$ is given by the formula

$$N(t) = N_0 e^{kt},$$

where $N(0) = N_0$ (read 'N nought') is the initial number of organisms and $k > 0$ is growth rate.

RADIOACTIVE DECAY:

The amount of a radioactive element at time t, $A(t)$ is given by the formula

$$A(t) = A_0 e^{kt},$$

where $A(0) = A_0$ is the initial amount of the element and $k < 0$ is the decay rate.

NEWTON'S LAW OF COOLING:

The temperature of an object at time t, $T(t)$ is given by the formula

$$T(t) = T_a + (T_0 - T_a)\, e^{-kt},$$

where $T(0) = T_0$ is the initial temperature of the object, T_a is the ambient temperature[1] and $k > 0$ is a constant.

LOGSTIC GROWTH:

If a population behaves according to the assumptions of logistic growth, the number of organisms at time t, $N(t)$ is given by

$$N(t) = \frac{L}{1 + Ce^{-kLt}},$$

where $N(0) = N_0$ is the initial population, L is the limiting population,[2] and C is a measure of how much room there is to grow given by

$$C = \frac{L}{N_0} - 1.$$

and $k > 0$ is a constant.

[1] That is, the temperature of the surroundings.

[2] That is, as $t \to \infty$, $N(t) \to L$

SAMPLE PROBLEMS

1. An account offers 5% interest, compounded monthly.

   How much will be in the account in 30 years if $450 is invested today?

   $$A(30) = 450\left(1 + \frac{0.05}{12}\right)^{12\cdot 30} \approx \$2010.48$$

2. An account offers 2.25% interest, compounded continuously.

   How much should Sally invest today if she wants $5000 in 10 years?

   $5000 = Pe^{0.0225\cdot 10}$ so $P = \frac{5000}{e^{0.0225\cdot 10}} = 5000e^{-0.225} \approx \$3992.58$

3. Skippy invests in an account offers 3.25% interest.

   (a) Find the doubling time if the interest is compounded *monthly*.

   $2P = P\left(1 + \frac{0.0325}{12}\right)^{12t}$. Dividing by P gives $\left(1 + \frac{0.0325}{12}\right)^{12t} = 2$.

   Taking natural logs, we get $\ln\left(1 + \frac{0.0325}{12}\right)^{12t} = \ln(2)$ or $12t\ln\left(1 + \frac{0.0325}{12}\right) = \ln(2)$.

   Hence, $t = \dfrac{\ln(2)}{12\ln\left(1 + \frac{0.0325}{12}\right)} \approx 21.36$ years.

   (b) Find the doubling time if the interest is compounded *continuously*.

   $2P = Pe^{0.0325t}$ so dividing by P gives $e^{0.0325t} = 2$.

   Taking natural logs, we get $\ln\left(e^{0.0325t}\right) = \ln(2)$ or $0.0325t = \ln(2)$.

   Hence, $t = \dfrac{\ln(2)}{0.0325} \approx 21.33$ years. (Not much different than part (a)!)

4. *Eludium Phosdax*, the so-called 'shaving cream atom' decays according to the formula $A(t) = A_0 e^{kt}$.

(a) If the half-life of this element is 25 minutes, find and interpret the decay constant, k.
HINT: The half-life of the material is how long it takes for half of the material to decay.
A_0 is the initial amount of the material, so after the half-life, there would be $\frac{1}{2}A_0$ of the material left.

Half life means it takes 25 minutes for half of the substance to decay (so half of it is left!)

Hence: $\frac{1}{2}A_0 = A_0 e^{25k}$. Dividing by A_0 gives $e^{25k} = \frac{1}{2}$.

Taking natural logs gives $\ln\left(e^{25k}\right) = \ln\left(\frac{1}{2}\right)$ or $25k = \ln\left(\frac{1}{2}\right)$.

Hence, $k = \dfrac{\ln(1/2)}{25} \approx -0.02772$. Our model is then: $A(t) = A_0 e^{-0.02772t}$.

(b) How long does it take for 90% of this material to decay?
Find an exact answer, then find an approximate the answer, rounded to the nearest minute.
HINT: If 90 % decays how much is left?

If 90% decays, only 10% is left. so we solve: $0.1A_0 = A_0 e^{-0.02772t}$.

Dividing by A_0, we get $e^{-0.02772t} = 0.1$.

Taking natural logs gives $\ln\left(e^{-0.02772t}\right) = \ln(0.1)$ or $-0.02772t = \ln(0.1)$.

Hence, $t = \dfrac{\ln(0.1)}{-0.02772} \approx 83$ minutes.

5. Newton's Law of Cooling states that the temperature T of an object at time t is given by:

$$T(t) = T_a + (T_0 - T_a)\, e^{-kt},$$

where $T(0) = T_0$ is the initial temperature of the object, and T_a is the ambient temperature.[3]

Suppose a piping hot cup of coffee (at 180°F) is served in a room that is 72°.

After 10 minutes, the coffee is 155°F.

(a) If the coffee cools according to Newton's Law of Cooling, find a formula for the temperature of the coffee in degrees Fahrenheit, $T(t)$, as a function of the number of minutes it has been left to cool, t.

Here, $T_0 = 180$, $T_a = 72$, so $T(t) = 72 + (180 - 72)\, e^{-kt}$ or $T(t) = 72 + 108e^{-kt}$.

After 10 minutes, the coffee is 155°F means $T(10) = 155$. Hence: $72 + 108e^{-10k} = 155$.

Solving this equation for k gives $108e^{-10k} = 83$ or $e^{-10k} = \frac{83}{108}$.

Taking natural logs gives $\ln\left(e^{-10k}\right) = \ln\left(\frac{83}{108}\right)$ or $-10k = \ln\left(\frac{83}{108}\right)$.

Hence, $k = -\frac{\ln(83/108)}{10} \approx 0.02633$.

So we have $T(t) = 72 + 108e^{-0.02633t}$

(b) How long does take for the coffee to cool to 120°F ?

To see how long it takes for the coffee to cool to 120°F means solving $T(t) = 72 + 108e^{-0.02633t} = 120$.

From $72 + 108e^{-0.02633t} = 120$ we get $108e^{-0.02633t} = 48$ or $e^{-0.02633t} = \frac{48}{108} = \frac{4}{9}$.

Taking the natural log, we get $-0.02633t = \ln\left(\frac{4}{9}\right)$, so $t = -\frac{1}{0.02633}\ln\left(\frac{4}{9}\right) \approx 31$ minutes.

(c) Find and interpret the horizontal asymptote to the graph of $y = T(t)$.

As $t \to \infty$, $-0.02633t \to -\infty$ so $e^{-0.02633t} \to 0$. Hence, $\lim_{t\to\infty} T(t) = 72 + 108(0) = 72$.

Hence, the horizontal asymptote to the graph of $y = T(t)$ is $y = 72$.

This means as time goes by, the coffee will cool to room temperature, 72°F

[3] That is, the temperature of the surroundings.

6. The population of Sasquatch, P, on Roskos Acres t years after 2003 is given by:

$$P(t) = \frac{150}{2 + 73e^{-0.5t}}, \quad t \geq 0$$

(a) Find and interpret $P(0)$.

$P(0) = \frac{150}{2+73e^{0}} = 2$. This means in 2003, there were just 2 Sasquatch in Roskos Acres.

(b) What is the population in 2007?

Since 2007 is 4 years after 2003, we find $P(4) \approx 12.62$ so between 12 and 13 Sasquatch.

(c) Solve $P(t) = 50$ and interpret.

Solve $P(t) = 50$ means solve $\frac{150}{2+73e^{-0.5t}} = 50$.

Clearing denominators, we get $150 = 50\left(2 + 73e^{-0.5t}\right)$ or $150 = 100 + 3650e^{-0.5t}$.

Hence, $3650e^{-0.5t} = 50$ so $e^{-0.5t} = \frac{50}{3650} = \frac{1}{73}$.

Taking natural logs we get $\ln\left(e^{-0.5t}\right) = \ln\left(\frac{1}{73}\right)$ or $-0.5t = \ln\left(\frac{1}{73}\right)$.

Hence, $t = -2\ln\left(\frac{1}{73}\right) = 2\ln(73) \approx 8.6$.

The population will reach 50 Sasquatch sometime during the year 2011.

(d) Find and interpret the horizontal asymptote of the graph of $y = P(t)$.

As $t \to \infty$, $-0.5t \to -\infty$, so $e^{-0.5t} \to 0$. Hence, $\lim_{t\to\infty} P(t) = \dfrac{150}{2+0} = 75$.

Hence, the horizontal asymptote of the graph of $y = P(t)$ is $y = 75$.

This means as time goes by, the Sasquatch Population will approach 75 Sasquatch.

CHALLENGE: Show that continuously compounded interest is the limit of compounded interest as the number of compounding periods per year approaches infinity. That is, show:

$$\lim_{x\to\infty} P\left(1 + \frac{r}{x}\right)^{xt} = Pe^{rt}$$

HINT: This limit requires L'Hopital's Rule...